



Angles with Rational Tangents

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4. Discussion. By a slightly more complicated argument we could prove that, for some positive constant c ,

$$f(x) < x \exp \{ -c(\log x)^{1/2} \};$$

but the true order of $f(x)$ seems to be considerably smaller. As far as I know, the only estimate for $f(x)$ from below is

$$f(x) > C \log x,$$

which is due to Lehmer.

Added later. As far as I know the question of the existence of even numbers satisfying $2^n \equiv 2 \pmod{n}$ has not been considered. Except for the trivial case $n=2$, I have not succeeded in finding any such even numbers.* By the method of this paper it is easy to see that their number $\leq x$ is certainly less than $x \exp \{ -\frac{1}{3} (\log x)^{1/4} \}$.

References

1. D. H. Lehmer, On the Converse of Fermat's Theorem, I, II, this MONTHLY, vol. 43, 1936, pp. 347-354; vol. 56, 1949, pp. 300-309. These papers contain references to other work on almost primes.
2. P. Erdős, On the Converse of Fermat's Theorem, this MONTHLY, vol. 56, 1949, pp. 623-624.

ANGLES WITH RATIONAL TANGENTS**

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1. Introduction. The purpose of this note is to show that the class of angles having rational tangents, and the class of angles which are rational multiples of π , intersect only in the obvious cases.

2. Theorem. We shall establish the following result.

THEOREM. *If x is a rational multiple of π , and $\tan x$ is rational, then x is an integral multiple of $\pi/4$.*

3. Proof. Let $\tan x = q/p$. The theorem is trivially satisfied if $q=0$ or if $|p| = |q|$. Further, without loss of generality, x may be restricted to the first quadrant, so that p and q may be assumed positive, integral, unequal, and coprime.

If $x = m\pi/n$, then $e^{inx} = e^{-inx} = \pm 1$, or

$$(\cos x + i \sin x)^n = (\cos x - i \sin x)^n,$$

and

$$(p + iq)^n / (p^2 + q^2)^{n/2} = (p - iq)^n / (p^2 + q^2)^{n/2}.$$

* Added still later: Lehmer has just informed me that $2^{161038} \equiv 2 \pmod{(161038)}$, $(161038 = 2 \cdot 73 \cdot 1103)$.

** Revised by J. D. Swift.

Thus

$$\begin{aligned} (p - iq)^n &= (p + iq)^n = (p - iq + 2iq)^n \\ &= (p - iq)^n + \binom{n}{1} (p - iq)^{n-1} 2iq + \dots \\ &\quad + \binom{n}{n-1} (p - iq) (2iq)^{n-1} + (2iq)^n. \end{aligned}$$

Therefore $(p - iq)$ divides $(2qi)^n$, and $p^2 + q^2$ divides $(2q)^{2n}$. Similarly, $p^2 + q^2$ divides $(2p)^{2n}$ and therefore divides $(2^{2n}p^{2n}, 2^{2n}q^{2n}) = 2^{2n}$. Then $p^2 + q^2 = 2^k$; but this is possible in positive coprime integers only when $p = q = k = 1$, a contradiction.

4. Corollary. By writing $\tan nx$ as a rational function in terms of $\tan x$, the reader may verify the following corollary.

COROLLARY: *The equations*

$$\sum_{k=0}^a (-1)^k \binom{n}{2k+1} x^{2k} = 0, \quad a = \left\lfloor \frac{n-1}{2} \right\rfloor, \quad n > 2, n \neq 4,$$

and

$$\sum_{k=0}^b (-1)^k \binom{n}{2k} x^{2k} = 0, \quad b = \left\lfloor \frac{n}{2} \right\rfloor, \quad n > 2,$$

have no rational roots.

A GENERALIZATION OF GAUSS' LEMMA*

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1. Introduction. We say that a number *belongs to the first half* modulo a number n if it is congruent mod n to one of the numbers $1, 2, \dots, [(n-1)/2]$, and that it *belongs to the second half* modulo n if it is congruent mod n to one of the numbers $[n/2] + 1, [n/2] + 2, \dots, n - 1$. The well-known Gauss' lemma can then be stated as follows:

A number A is a quadratic residue modulo an odd prime p if and only if an even number of the terms

$$(1) \quad A, 2A, 3A, \dots, (p-1)A/2$$

belongs to the second half mod p .

2. Theorem. In this note we shall prove the following result.

THEOREM: *If p is an odd prime and A is odd, then the number of terms in the sequence (1) which belong to the second half mod p is equal to the number of terms which belong to the second half mod $2p$.*

* Translated and revised by E. G. Straus.